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AVAILABILITY EQUATIONS FOR REDUNDANT SYSTEMS, BOTH SINGLE AND MULTIPLE REPAIR CAPABILITY

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This memo documents derivations of availability equations for redundant systems not readily available in current literature.

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Reliability papers, technical reports, and textbooks almost universally document the derivation of availability equations for two unit full-on parallel redundant systems with the assumption of "multiple repair" crews or facilities. Multiple repair is the case where there are as many repair crews available as there are possible units to fail. Some reliability literature also derive the availability equations for two-unit standby redundant systems with the multiple repair assumption. Literature is scarce, however, for two unit full-on and standby parallel systems with single repair capability (i.e. only one repair crew is available and hence, failed units can only be repaired one at a time); it is virtually non-existent with respect to models of systems made up of more than two redundant units.

This paper documents the derivation of availability models for two unit and three unit full-on and standby parallel systems with single repair capabilities. Also included are derivations for three unit full-on and standby systems with single and multiple repair capability. In all cases, each unit will be assumed to have a failure rate = λ and a repair rate = μ.

Case 1: The system consists of two units both fully operational whenever possible and has single repair capability. The system is capable of performing its mission when only one unit is operational. There are three possible states for the system to be in at any given time:

State 2 - both units operating

1 - one of the units has failed, other unit operating

0 - both units have failed.

Using a Markovian approach and letting $P_2(t)$, $P_1(t)$, and $P_o(t)$ represent the probability of being in states 2, 1, and 0 respectively at time t yields:

$$P_2(t+\Delta t) = P_2(t)\,(1-2\lambda\Delta t) + P_1(t)\mu\Delta t$$

$$P_1(t+\Delta t) = P_2(t)\,(2\lambda\Delta t) + P_1(t)\,(1-(\lambda+\mu)\Delta t) + P_o(t)\mu\Delta t$$

$$P_o(t+\Delta t) = P_1(t)\,(\lambda\Delta t) + P_o(t)\,(1-\mu\Delta t)$$

Expanding and rearranging terms using the definition of a derivative:

$$\dot{P}_2(t) = -2\lambda P_2(t) + \mu\, P_1(t)$$

$$\dot{P}_1(t) = 2\lambda P_2(t) - (\lambda+\mu)\, P_1(t) + \mu P_o(t)$$

$$\dot{P}_o(t) = \lambda P_1(t) - \mu\, P_o(t)$$

Taking Laplace transforms and realizing that:

$P_2(0) = 1$, $P_1(0) = 0 = P_o(0)$ Assume all units operative at t=0

$$(S+2\lambda)\ P_2(S)\ -\ \mu\ P_1(S) = 1$$

$$2\lambda\ P_2(S)\ -\ (S+\lambda+\mu)\ P_1(S) + \mu\ P_0(S) = 0$$

$$-\ \lambda\ P_1(S) + (S+\mu)\ P_0(S) = 0$$

Solving these simultaneous equations for P_0 results in:

$$P_0(S) = \frac{2\lambda^2}{S(S^2 + S(3\lambda+2\mu) + (2\lambda^2 + 2\lambda\mu + \mu^2))}$$

The denominator must first be factored and then partial fraction expansion accomplished to put the equation in a form to find the inverse Laplace transform.

This yields:

$$P_0(S) = \frac{2\lambda^2}{S(S-S_1)(S-S_2)}$$

where $S_1, S_2 = \frac{1}{2}\left[-(3\lambda+2\mu) \pm \sqrt{\lambda^2+4\lambda\mu}\right]$ and are the roots of $S^2+S(3\lambda+2\mu)+(2\lambda^2+2\lambda\mu+\mu^2)$

$$= 2\lambda^2 \left[\frac{1}{S_1S_2S} - \frac{1}{S_1(S_2-S_1)(S-S_1)} + \frac{1}{S_2(S_2-S_1)(S-S_2)}\right]$$

Finding the inverse transform yields:

$$P_0(t) = 2\lambda^2 \left[\frac{1}{S_1S_2} - \frac{e^{S_1t}}{S_1(S_2-S_1)} + \frac{e^{S_2t}}{S_2(S_2-S_1)} \right]$$

$$= 2\lambda^2 \left[\frac{(S_2-S_1) - S_2e^{S_1t} + S_1e^{S_2t}}{S_1S_2(S_2-S_1)} \right]$$

So availability then is:

$$A = 1-P_0(t) = \frac{S_1S_2(S_2-S_1) - 2\lambda^2 \left[S_2-S_1-S_2e^{S_1t} + S_1e^{S_2t} \right]}{S_1S_2(S_2-S_1)}$$

$$= \frac{S_1S_2-2\lambda^2}{S_1S_2} + \frac{2\lambda^2 \left[S_2e^{S_1t} - S_1e^{S_2t} \right]}{S_1S_2(S_2-S_1)}$$

$$= \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu+\mu^2} + \frac{2\lambda^2 \left[S_2e^{S_1t} - S_1e^{S_2t} \right]}{S_1S_2(S_2-S_1)}$$

Case 2: The system consists of two units with one unit operating and one in a standby state and has single repair capability. When a failure occurs, perfect switching is assumed to connect the standby unit and system operation is not affected. The standby unit is assumed to have a failure rate equal to 0, the operating unit is assumed to have a failure rate equal to λ. A unit under repair is assumed to have a repair rate = μ. The three possible states are the same as in Case 1.

Equations of state:

$$P_2(t+\Delta t) = P_2(t)\ (1-\lambda\Delta t) + P_1(t)\mu\Delta t$$

$$P_1(t+\Delta t) = P_2(t)\ \lambda\Delta t + P_1(t)\ (1-(\lambda+\mu)\Delta t) + P_o(t)\mu\Delta t$$

$$P_o(t+\Delta t) = P_1(t)\lambda\Delta t + P_o(t)\ (1-\mu\Delta t)$$

$$\dot{P}_2(t) = -\lambda P_2(t) + \mu P_1(t)$$

$$\dot{P}_1(t) = \lambda P_2(t) - (\lambda+\mu)P_1(t) + \mu P_o(t)$$

$$\dot{P}_o(t) = \lambda P_1(t) - \mu\ P_o(t)$$

Taking Laplace transforms:

$$(S+\lambda)\ P_2(S)\ -\ \mu\ P_1(S)\ =\ 1$$

$$\lambda\ P_2(S)\ -\ (S+\lambda+\mu)\ P_1(S)\ +\ \mu\ P_0(S)\ =\ 0$$

$$-\lambda\ P_1(S)\ +\ (S+\mu)\ P_0(S)\ =\ 0$$

Solution for $P_0(S)$ after solving the above equation set simultaneously:

$$P_0(S) = \frac{\lambda^2}{S(S-S_1)(S-S_2)}$$

where $S_1,\ S_2 = -(\lambda+\mu) \pm \sqrt{\lambda\mu}$

These are the roots of $S^2+S(2\lambda+2\mu)+(\lambda^2+\lambda\mu+\mu^2)$

Having the denominator factored enables partial fraction expansion and hence finding the inverse transform.

The Inverse transform of the above results in:

$$P_0(t) = \lambda^2 \left[\frac{S_2-S_1\ -\ S_2e^{S_1t}\ +\ S_1e^{S_2t}}{S_1S_2(S_2-S_1)} \right]$$

Availability:

$$A(t) = 1-P_0(t) \quad = \quad \frac{\lambda\mu+\mu^2}{\lambda^2+\lambda\mu+\mu^2} + \frac{\lambda^2\left[S_2e^{S_1t} - S_1e^{S_2t}\right]}{S_1S_2(S_2-S_1)}$$

For three unit systems, there are 4 possible states:

3 - all units operable

2 - 2 units operable, 1 unit failed

1 - 1 unit operable, 2 units failed

0 - all units failed

Case 3: Three different three unit systems will be investigated. Three unit full-on parallel operation with single repair.

State Equations:

$$P_3(t+\Delta t) = P_3(t)(1-3\lambda\Delta t) + P_2(t)\mu\Delta t$$

$$P_2(t+\Delta t) = P_3(t)\ 3\lambda\Delta t + P_2(t)(1-(2\lambda+\mu)\Delta t) + P_1(t)\mu\Delta t$$

$$P_1(t+\Delta t) = P_2(t)\ 2\lambda\Delta t + P_1(t)(1-(\lambda+\mu)\Delta t) + P_0(t)\mu\Delta t$$

$$P_0(t+\Delta t) = P_1(t)\ \lambda\Delta t + P_0(t)(1-\mu\Delta t)$$

$$\dot{P}_3(t) = -3\lambda P_3(t) + \mu P_2(t)$$

$$\dot{P}_2(t) = 3\lambda P_3(t) - (2\lambda+\mu) P_2(t) + \mu P_1(t)$$

$$\dot{P}_1(t) = 2\lambda P_2(t) - (\lambda+\mu) P_1(t) + \mu P_o(t)$$

$$\dot{P}_o(t) = \lambda P_1(t) - \mu P_o(t)$$

Taking Laplace Transforms:

$$(S+3\lambda) P_3(S) - \mu P_2(S) = 1$$

$$3\lambda P_3(S) - (S+2\lambda+\mu) P_2(S) + \mu P_1(S) = 0$$

$$2\lambda P_2(S) - (S+\lambda+\mu) P_1(S) + \mu P_o(S) = 0$$

$$\lambda P_1(S) - (S+\mu) P_o(S) = 0$$

Simultaneous Solution of the above for $P_o(S)$ results in:

$$P_o(S) = \frac{-6\lambda^3}{S(S-S_1)(S-S_2)(S-S_3)}$$ where S_1, S_2, S_3 are roots of

$$S^3+S^2(6\lambda+3\mu) + S(11\lambda^2+9\lambda\mu+3\mu^2) + (6\lambda^3+3\lambda\mu^2+\mu^3+6\lambda^2\mu)$$

To find values for S_1, S_2, and S_3 note the cubic equation is of the form:

$$S^3 + pS^2 + qS + r = 0$$

Where: $p = 6\lambda + 3\mu$

$q = 11\lambda^2 + 9\lambda\mu + 3\mu^2$

$r = 6\lambda^3 + 6\lambda^2\mu + 3\lambda\mu^2 + \mu^3$

Substituting $(x - p/3)$ for S reduces the equation to:

$$x^3 + ax + b = 0$$

Where: $a = 1/3\ (3q - p^2)$

$b = 1/27\ (2p^3 - 9pq + 27r)$

Since $\frac{b^2}{4} + \frac{a^3}{27} < 0$ there are 3 distinct real roots.

Determine the value of the θ in the expression $\cos\theta = -b \div 2\sqrt{\frac{-a^3}{27}}$

The roots x_1, x_2, x_3 will have the values: $x_1 = 2\sqrt{\frac{-a}{3}}\ \cos\frac{\theta}{3}$

$$x_2 = 2\sqrt{\frac{-a}{3}}\cos\left[\frac{\theta}{3} + \frac{2\pi}{3}\right]$$

$$x_3 = 2\sqrt{\frac{-a}{3}}\cos\left[\frac{\theta}{3} + \frac{4\pi}{3}\right]$$

S_1, S_2, S_3 are then calculated from:

$$S_1 = x_1 - \frac{p}{3}$$

$$S_2 = x_2 - \frac{p}{3}$$

$$S_3 = x_3 - \frac{p}{3}$$

The inverse transform for P_o is:

$$P_o(t) = 6\lambda^3 \; x$$

$$\left[\frac{(S_1-S_3)(S_1-S_2)(S_2-S_3) - S_2S_3(S_2-S_3)e^{S_1t} + S_1S_3(S_1-S_3)e^{S_2t} - S_1S_2(S_1-S_2)e^{S_3t}}{S_1S_2S_3(S_1-S_3)(S_1-S_2)(S_2-S_3)}\right]$$

Availability:

$$A(t) = 1 - P_0(t)$$

$$= \frac{\mu^3+3\lambda\mu^2+6\lambda^2\mu}{\mu^3+3\lambda\mu^2+6\lambda^2\mu+6\lambda^3} +$$

$$\frac{6\lambda^3\left[S_2S_3(S_2-S_3)e^{S_1t} - S_1S_3(S_1-S_3)e^{S_2t} + S_1S_2(S_1-S_2)e^{S_3t}\right]}{S_1S_2S_3(S_1-S_3)(S_1-S_2)(S_2-S_3)}$$

Case 4: Three unit full-on parallel operation with multiple repair.

State equations:

$$P_3(t+\Delta t) = P_3(t)(1-3\lambda\Delta t)+P_2(t)\mu\Delta t$$

$$P_2(t+\Delta t) = P_3(t)\ 3\lambda\Delta t + P_2(t)(1-(2\lambda+\mu)\Delta t) + P_1(t)2\mu\Delta t$$

$$P_1(t+\Delta t) = P_2(t)\ 2\lambda\Delta t + P_1(t)(1-(\lambda+2\mu)\Delta t) + P_0(t)3\mu\Delta t$$

$$P_0(t+\Delta t) = P_1(t)\ \lambda\Delta t + P_0(t)(1-3\mu\Delta t)$$

$$\dot{P}_3(t) = -3\lambda\ P_3(t) + \mu\ P_2(t)$$

$$\dot{P}_2(t) = 3\lambda\ P_3(t) - (2\lambda+\mu)\ P_2(t) + 2\mu\ P_1(t)$$

$$\dot{P}_1(t) = 2\lambda\ P_2(t) - (\lambda+2\mu)\ P_1(t) + 3\mu\ P_0(t)$$

$$\dot{P}_0(t) = \lambda\ P_1(t) - 3\mu\ P_0(t)$$

Taking Laplace Transforms:

$$(S+3\lambda)\ P_3(S) - \mu\ P_2(S) = 1$$

$$3\lambda\ P_3(S) - (S+2\lambda+\mu)\ P_2(S) + 2\mu\ P_1(s) = 0$$

$$2\lambda\ P_2(S) - (S+\lambda+2\mu)\ P_1(S) + 3\mu\ P_0(s) = 0$$

$$\lambda\ P_1(S) - (S+3\mu)\ P_0(S) = 0$$

Simultaneous Solution of the above for $P_o(S)$:

$$P_o(S) = \frac{-6\lambda^3}{S(S-S_1)(S-S_2)(S-S_3)}$$

Where S_1, S_2, S_3 are roots of $S^3 + S^2\,(6\lambda+6\mu)+S(11(\lambda+\mu)^2)+6(\lambda+\mu)^3$

$$S_1 = -(\lambda + \mu)$$
$$S_2 = -2(\lambda + \mu)$$
$$S_3 = -3(\lambda + \mu)$$

The Inverse transform is the same as for Case 3 except for the values of S_1, S_2, and S_3. Availability then is:

$$A(t) = 1 - P_o(t)$$

$$= \frac{\mu^3+3\lambda\mu^2+3\lambda^2\mu}{(\lambda+\mu)^3} +$$

$$\frac{6\lambda^3\left[S_2S_3(S_2-S_3)e^{S_1t} - S_1S_3(S_1-S_3)e^{S_2t} + S_1S_2(S_1-S_2)e^{S_3t}\right]}{S_1S_2S_3(S_1-S_3)(S_1-S_2)(S_2-S_3)}$$

Derivations for the standby parallel system single or multiple repair are found in the same manner as above.

The results are:

Case 5: 3 unit standby with single repair:

A(t)

$$= \frac{\mu^3+\lambda\mu^2+\lambda^2\mu}{\mu^3+\lambda\mu^2+\lambda^2\mu+\lambda^3} + \frac{\lambda^3\left[S_2S_3(S_2-S_3)e^{S_1t} - S_1S_3(S_1-S_3)e^{S_2t} + S_1S_2(S_1-S_2)e^{S_3t}\right]}{S_1S_2S_3(S_1-S_2)(S_1-S_3)(S_2-S_3)}$$

Where S_1, S_2, S_3 are roots of $S^3+S^2(3\lambda+3\mu)+S(3\lambda^2+4\lambda\mu+3\mu^2)+(\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3)$

$$S_1 = -(\lambda + \mu)$$

$$S_2, S_3 = -(\lambda + \mu) \pm\sqrt{2\lambda\mu}$$

Case 6: Three unit standby with multiple repair:

$A(t)$

$$= \frac{6\mu^3+6\lambda\mu^2+3\lambda^2\mu}{6\mu^3+6\lambda\mu^2+3\lambda^2\mu+\lambda^3} +$$

$$\frac{\lambda^3\left[S_2S_3(S_2-S_3)e^{S_1t} - S_1S_3(S_1-S_3)e^{S_2t} + S_1S_2(S_1-S_2)e^{S_3t}\right]}{S_1S_2S_3(S_1-S_2)(S_1-S_3)(S_2-S_3)}$$

Where S_1, S_2, S_3 are roots of $S^3+S^2(3\lambda+6\mu)+S(3\lambda^2+9\lambda\mu+11\mu^2)+(\lambda^3+3\lambda^2\mu+6\lambda\mu^2+6\mu^3)$

To find values for S_1, S_2, and S_3 note the cubic equation is of the form:

$$S^3 + pS^2 + qS + r = 0$$

Where: $p = 3\lambda + 6\mu$

$q = 3\lambda^2 + 9\lambda\mu + 11\mu^2$

$r = \lambda^3 + 3\lambda^2\mu + 6\lambda\mu^2 + 6\mu^3$

Substituting $(x - p/3)$ for S reduces the equation to:

$$x^3 + ax + b = 0$$

Where: $a = 1/3\ (3q - p^2)$

$b = 1/27\ (2p^3 - 9pq + 27r)$

Since $\frac{b^2}{4} + \frac{a^3}{27} < 0$ there are 3 distinct real roots.

Determine the value of the θ in the expression $\cos\theta = -b \div 2\sqrt{\frac{-a^3}{27}}$

The roots x_1, x_2, x_3 will have the values: $x_1 = 2\sqrt{\frac{-a}{3}}\cos\frac{\theta}{3}$

$$x_2 = 2\sqrt{\frac{-a}{3}}\cos\left[\frac{\theta}{3} + \frac{2\pi}{3}\right]$$

$$x_3 = 2\sqrt{\frac{-a}{3}}\cos\left[\frac{\theta}{3} + \frac{4\pi}{3}\right]$$

S_1, S_2, S_3 are then calculated from:

$$S_1 = x_1 - \frac{p}{3}$$

$$S_2 = x_2 - \frac{p}{3}$$

$$S_3 = x_3 - \frac{p}{3}$$

Documentation for the derivation of all the availability equations for redundant systems listed in Section 10 of MIL-HDBK-338, Electronic Reliability Design Handbook, could not be found. The above derivations were accomplished to correct this deficiency and to check the accuracy of those equations.

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